NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 2–7)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 8–14)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Which diagram best represents the solution set of $x^2 - 2x - 3 \geq 0$?

A.

B.

C.

D.

2 Given $f(x) = 1 + \sqrt{x}$, what are the domain and range of $f^{-1}(x)$?

A. $x \geq 0, \quad y \geq 0$

B. $x \geq 0, \quad y \geq 1$

C. $x \geq 1, \quad y \geq 0$

D. $x \geq 1, \quad y \geq 1$

3 Which of the following is an anti-derivative of $\dfrac{1}{4x^2 + 1}$?

A. $2\tan^{-1}\left(\dfrac{x}{2}\right) + c$

B. $\dfrac{1}{2}\tan^{-1}\left(\dfrac{x}{2}\right) + c$

C. $2\tan^{-1}(2x) + c$

D. $\dfrac{1}{2}\tan^{-1}(2x) + c$

4 Maria starts at the origin and walks along all of the vector $2\underset{\sim}{i} + 3\underset{\sim}{j}$, then walks along all of the vector $3\underset{\sim}{i} - 2\underset{\sim}{j}$ and finally along all of the vector $4\underset{\sim}{i} - 3\underset{\sim}{j}$.

How far from the origin is she?

A. $\sqrt{77}$

B. $\sqrt{85}$

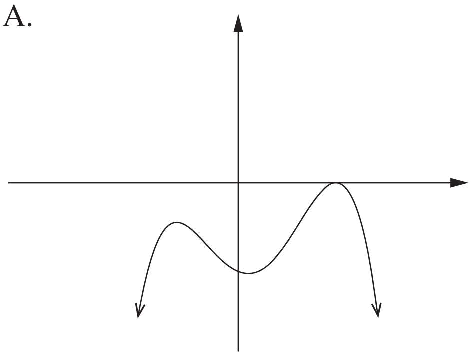
C. $2\sqrt{13} + \sqrt{5}$

D. $\sqrt{5} + \sqrt{7} + \sqrt{13}$

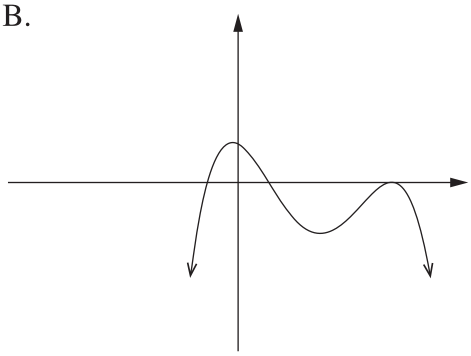
5 A monic polynomial $p(x)$ of degree 4 has one repeated zero of multiplicity 2 and is divisible by $x^2 + x + 1$.

Which of the following could be the graph of $p(x)$?

A.



B.



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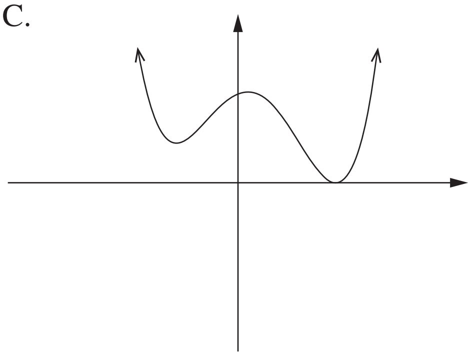
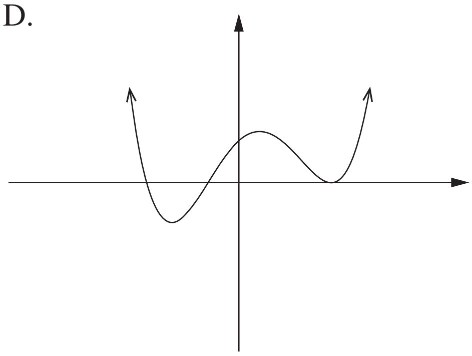
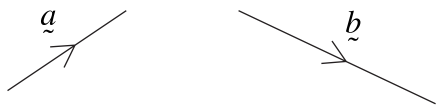


D.

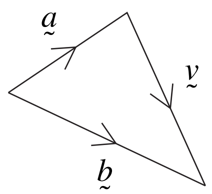


6 The vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are shown.

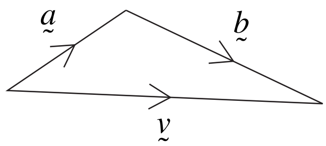


Which diagram below shows the vector $\underset{\sim}{v} = \underset{\sim}{a} - \underset{\sim}{b}$?

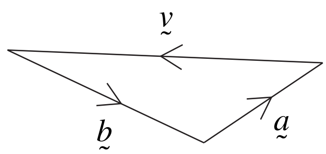
A.



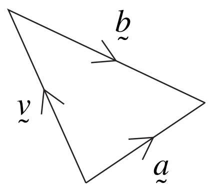
B.



C.

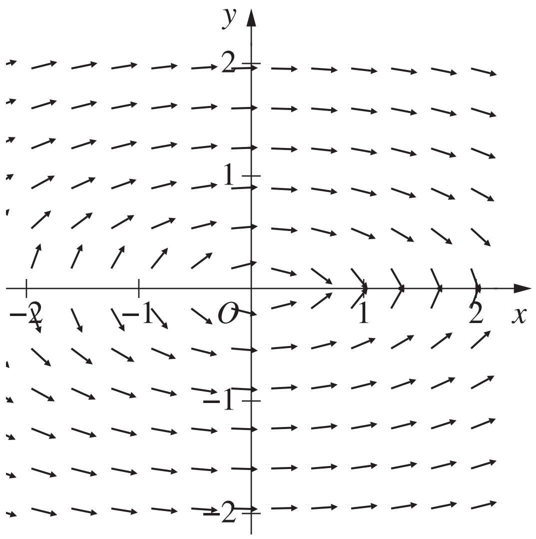


D.

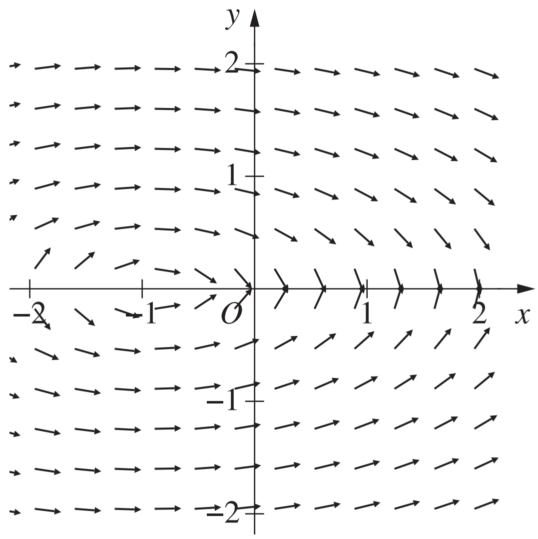


7 Which of the following best represents the direction field for the differential equation $\dfrac{dy}{dx} = -\dfrac{x}{4y}$?

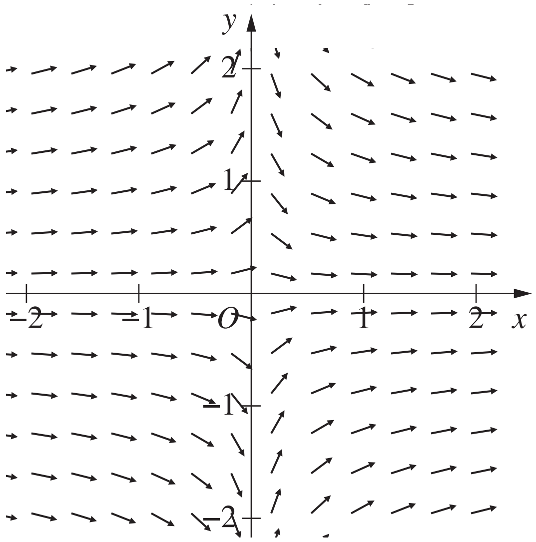
A.



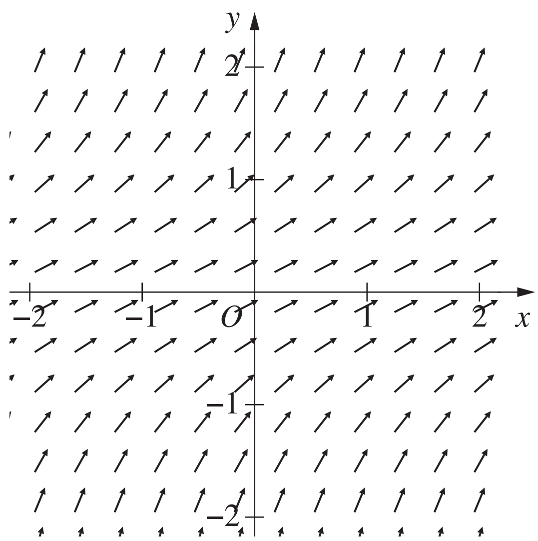
B.



C.



D.



8 Out of 10 contestants, six are to be selected for the final round of a competition. Four of those six will be placed 1st, 2nd, 3rd and 4th.

In how many ways can this process be carried out?

A. $\dfrac{10!}{6!4!}$

B. $\dfrac{10!}{6!}$

C. $\dfrac{10!}{4!2!}$

D. $\dfrac{10!}{4!4!}$

9 The projection of the vector $\begin{pmatrix} 6 \\ 7 \end{pmatrix}$ onto the line $y = 2x$ is $\begin{pmatrix} 4 \\ 8 \end{pmatrix}$.

The point (6, 7) is reflected in the line $y = 2x$ to a point A.

What is the position vector of the point A?

A. $\begin{pmatrix} 6 \\ 12 \end{pmatrix}$

B. $\begin{pmatrix} 2 \\ 9 \end{pmatrix}$

C. $\begin{pmatrix} -6 \\ 7 \end{pmatrix}$

D. $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

10 The quantities P, Q and R are connected by the related rates,

$$\frac{dR}{dt} = -k^2$$

$$\frac{dP}{dt} = -l^2 \times \frac{dR}{dt}$$

$$\frac{dP}{dt} = m^2 \times \frac{dQ}{dt}$$

where k, l and m are non-zero constants.

Which of the following statements is true?

A. P is increasing and Q is increasing

B. P is increasing and Q is decreasing

C. P is decreasing and Q is increasing

D. P is decreasing and Q is decreasing

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet

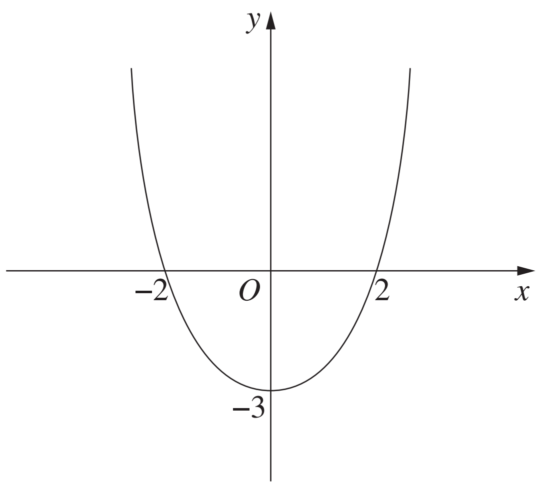
(a) Let $P(x) = x^3 + 3x^2 - 13x + 6$.

(i) Show that $P(2) = 0$. **1**

(ii) Hence, factor the polynomial $P(x)$ as $A(x)B(x)$, where $B(x)$ is a quadratic polynomial. **2**

(b) For what value(s) of a are the vectors $\begin{pmatrix} a \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2a-3 \\ 2 \end{pmatrix}$ perpendicular? **3**

(c) The diagram shows the graph of $y = f(x)$. **3**



Sketch the graph of $y = \dfrac{1}{f(x)}$.

Question 11 continues on page 9

(d) By expressing $\sqrt{3}\sin x + 3\cos x$ in the form $A\sin(x + \alpha)$, solve $\sqrt{3}\sin x + 3\cos x = \sqrt{3}$, for $0 \le x \le 2\pi$. **4**

(e) Solve $\dfrac{dy}{dx} = e^{2y}$, finding x as a function of y. **2**

End of Question 11

Please turn over

Question 12 (14 marks) Use the Question 12 Writing Booklet

(a) Use the principle of mathematical induction to show that for all integers $n \geq 1$, **3**

$$1 \times 2 + 2 \times 5 + 3 \times 8 + \cdots + n(3n-1) = n^2(n+1).$$

(b) When a particular biased coin is tossed, the probability of obtaining a head is $\frac{3}{5}$.

This coin is tossed 100 times.

Let X be the random variable representing the number of heads obtained. This random variable will have a binomial distribution.

(i) Find the expected value, $E(X)$. **1**

(ii) By finding the variance, $\mathrm{Var}(X)$, show that the standard deviation of X is approximately 5. **1**

(iii) By using a normal approximation, find the approximate probability that X is between 55 and 65. **1**

(c) To complete a course, a student must choose and pass exactly three topics. **2**

There are eight topics from which to choose.

Last year 400 students completed the course.

Explain, using the pigeonhole principle, why at least eight students passed exactly the same three topics.

(d) Find $\displaystyle\int_0^{\frac{\pi}{2}} \cos 5x \sin 3x \, dx$. **3**

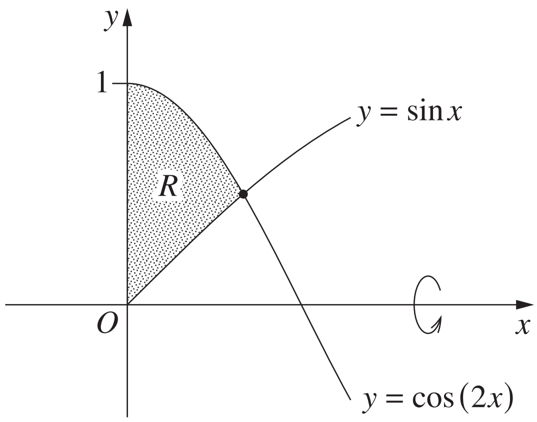
(e) Find the curve which satisfies the differential equation $\dfrac{dy}{dx} = -\dfrac{x}{y}$ and passes through the point $(1, 0)$. **3**

Question 13 (16 marks) Use the Question 13 Writing Booklet

(a) (i) Find $\dfrac{d}{d\theta}\left(\sin^3\theta\right)$. **1**

(ii) Use the substitution $x = \tan\theta$ to evaluate $\displaystyle\int_0^1 \frac{x^2}{\left(1+x^2\right)^{\frac{5}{2}}}\,dx$. **4**

(b) The region R is bounded by the y-axis, the graph of $y = \cos(2x)$ and the graph of $y = \sin x$, as shown in the diagram. **4**

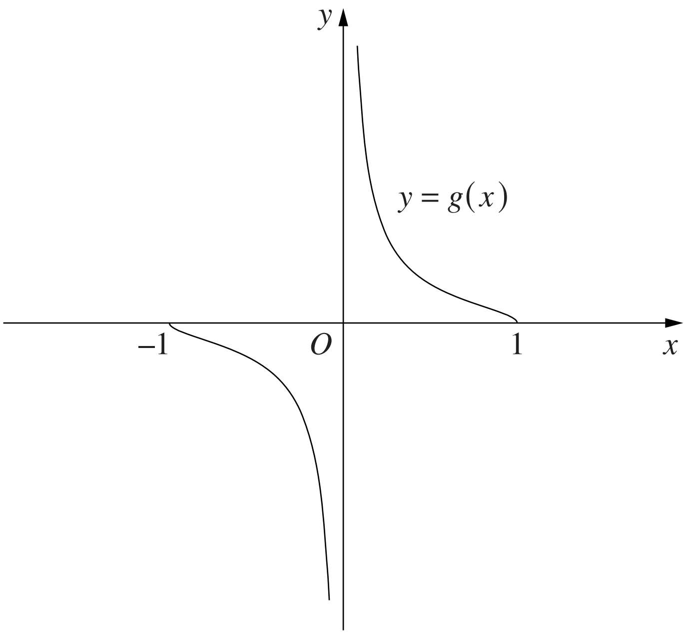


Find the volume of the solid of revolution formed when the region R is rotated about the x-axis.

Question 13 continues on page 12

(c) Suppose $f(x) = \tan\left(\cos^{-1}(x)\right)$ and $g(x) = \dfrac{\sqrt{1-x^2}}{x}$.

The graph of $y = g(x)$ is given.



(i) Show that $f'(x) = g'(x)$. **4**

(ii) Using part (i), or otherwise, show that $f(x) = g(x)$. **3**

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet

(a) (i) Use the identity $(1+x)^{2n} = (1+x)^n (1+x)^n$ **2**

to show that

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2,$$

where n is a positive integer.

(ii) A club has $2n$ members, with n women and n men. **2**

A group consisting of an even number $(0, 2, 4, \ldots, 2n)$ of members is chosen, with the number of men equal to the number of women.

Show, giving reasons, that the number of ways to do this is $\binom{2n}{n}$.

(iii) From the group chosen in part (ii), one of the men and one of the women are selected as leaders. **2**

Show, giving reasons, that the number of ways to choose the even number of people and then the leaders is

$$1^2\binom{n}{1}^2 + 2^2\binom{n}{2}^2 + \cdots + n^2\binom{n}{n}^2.$$

(iv) The process is now reversed so that the leaders, one man and one woman, are chosen first. The rest of the group is then selected, still made up of an equal number of women and men. **2**

By considering this reversed process and using part (ii), find a simple expression for the sum in part (iii).

Question 14 continues on page 14

Question 14 (continued)

(b) (i) Show that $\sin^3\theta - \frac{3}{4}\sin\theta + \frac{\sin(3\theta)}{4} = 0.$ **2**

(ii) By letting $x = 4\sin\theta$ in the cubic equation $x^3 - 12x + 8 = 0.$ **2**

Show that $\sin(3\theta) = \frac{1}{2}.$

(iii) Prove that $\sin^2\frac{\pi}{18} + \sin^2\frac{5\pi}{18} + \sin^2\frac{25\pi}{18} = \frac{3}{2}.$ **3**

End of paper

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Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}\left[2a + (n - 1)d\right] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1},\ r \neq 1$$

$$S = \frac{a}{1 - r},\ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

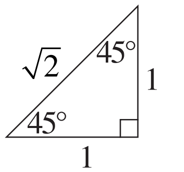
$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

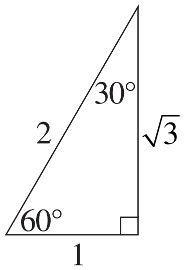


$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$



Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

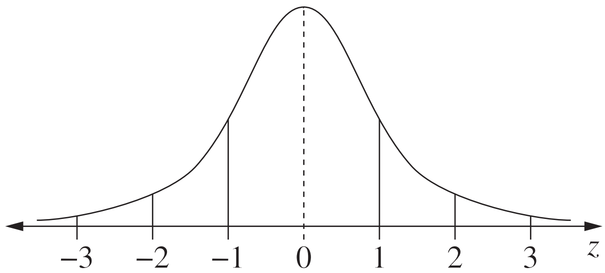
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \le r) = \int_a^r f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1-p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \quad P(X = x)$$

$$= \binom{n}{x} p^x (1-p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1-p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n\,dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\Big\{f(a) + f(b) + 2\big[f(x_1) + \cdots + f(x_{n-1})\big]\Big\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$$^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {^nC_r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$